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Author(s): Tapan Mitra
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Efficiency, Weak Value Maximality and Weak Value Optimality in a Multisector Model

TAPAN MITRA
State University of New York, Stony Brook

1. Introduction

An important problem in the theory of efficient allocation of resources over time, in an infinite horizon model, is to examine whether an efficient programme maximizes the present value of its consumption sequence, in the set of all feasible consumption sequences. A natural method of determining the present value of an efficient programme is to evaluate the consumption sequence at the competitive (intertemporal profit maximizing) prices, associated with it. The difficulty is that competitive prices associated with an efficient programme need not define a finite present value of consumption.

One way out of this "infinite-horizon problem" is to restrict the maximization to a certain subset of all feasible programmes, and this is the idea pursued by Malinvaud (1953). He associates with each (non-tight) efficient programme, a sequence of competitive prices at which the value of the consumption sequence of the efficient programme, truncated at any finite horizon T, is maximal among all feasible consumption sequences, truncated at the same T, and having identical consumption sequences as the efficient programme beyond T.

A more satisfactory route is to relax the concept of "maximization" itself, and this is the approach of Peleg and Yaari (1970), and of Cass and Yaari (1971). They associate with an efficient programme a sequence of competitive prices, at which it is "weakly value-maximal" among all feasible programmes. This means that an efficient programme cannot be overtaken, in terms of the value of consumption, by a finite (positive) amount, by any feasible programme.

The present investigation can be viewed as an extension of the second approach. It should be noted that the contribution of Cass and Yaari (1971) was confined to a one-good model. The analysis of Peleg and Yaari (1970) proceeds by making assumptions on the infinite-dimensional space of consumption sequences. Thus, even though in principle their analysis is applicable to multisector models, their assumptions are rather difficult to verify in such models (see particularly (Y.5) and (Y.6) on pages 72 and 78 of their paper). The first purpose of this paper is to establish a generalized version of the Cass–Yaari theorem in the multisector neoclassical model of Dorfman–Samuelson–Solow (1958).

The model has been specified in detail in Mitra (1976). The reader is asked to refer to that paper for the assumptions used (which amount to the assumption of a smooth neoclassical technology, with a differentiable frontier in the interior of R_+^{2m-1}), definitions of concepts, and statements and proofs of results. For ease of exposition we recall just a few facts from Mitra (1976). Associated with any interior programme $(\bar{x}, \bar{y}, i)$ is a current price sequence $(\bar{q}_t)$, and a discounted price sequence $(\bar{p}_t)$. These price sequences are defined "technologically" by the marginal rates of transformation between the m-th output and the remaining $m-1$ outputs, and between the m inputs and the m-th output. By our assumptions on the technology, these prices are strictly positive for all $t \geqq 0$. Furthermore, if $(\bar{x}, \bar{y}, \bar{c})$ is competitive then $(\bar{x}_t, \bar{y}_{t+1})$ maximizes intertemporal profit at the price sequence $(\bar{p}_t)$ at each date; that is,

$$\bar{p}_{t+1}\bar{y}_{t+1} - \bar{p}_t\bar{x}_t \geqq \bar{p}_{t+1}y - \bar{p}_t x \quad \text{for } t \geqq 0 \tag{1}$$

for every input–output pair (x, y) in the technology set. Thus $(\bar{p}_t)$ can also be called a sequence of competitive or intertemporal profit maximising prices.

Some additional concepts used in this paper are defined below. An interior programme $(\bar{x}, \bar{y}, \bar{c})$ is weakly value maximal if

$$\liminf_{T\to\infty} \sum_{t=1}^{T} \bar{p}_t(c_t - \bar{c}_t) \leqq 0 \tag{2}$$

for every feasible programme (x, y, c). It is weakly value optimal if

$$\limsup_{T\to\infty} \sum_{t=1}^{T} \bar{p}_t(c_t - \bar{c}_t) \leqq 0 \tag{3}$$

for every feasible programme (x, y, c). It is regular interior if

$$\inf_{t\geqq 1} \bar{c}_t^i > 0 \quad \text{for } i = 1, \ldots, m. \tag{4}$$

It has a bounded discounted price sequence if

$$\sup_{t\geqq 1} \bar{p}_t^i < \infty \quad \text{for } i = 1, \ldots, m. \tag{5}$$

The generalized version of the Cass–Yaari result is given by

Theorem 1. *Under* (A.1)–(A.4), *an interior program* $(\bar{x}, \bar{y}, \bar{c})$ *is efficient if and only if it is weakly value maximal.*

The proof (see Section 2) is quite different from that used by Cass and Yaari (1971). In fact, it is difficult to see how their method of proof can be generalized, as the problem of feasibility is much more intricate in the multisectoral case. We use the methods employed in Mitra (1976) to establish a complete characterization of efficiency in such a model. This has the advantage of revealing the close formal relationship between the problem of obtaining a complete characterization of efficiency, and the problem of proving a weak value maximality property of efficient programmes.

It is worth mentioning that Theorem 1 is restrictive in two respects. First, we restrict ourselves to the case of a stationary technology. This is because, in our proof, we need to use the fact that the inputs are all uniformly bounded above. This is not ensured when there is sufficient technical progress. Second, we restrict the characterization only to interior programs. This is to avoid cases where the derivatives of the production locus become unbounded at zero. These are essential aspects of the method of proof that is followed. Whether one can devise an alternative method that can relax these two restrictions remains an open question.

It is known that an efficient programme need not be weakly value optimal. That is, it need not "catch up" in terms of the value of its consumption to every feasible programme. This was shown by an example in Cass–Yaari ((1971), p. 335). It would be interesting to separate the class of efficient programmes which are weakly value optimal from those which are not.

In order to look for the proper criterion, an appropriate starting point is the Cass-Yaari example. This involved the phenomenon that the competitive prices associated with the constructed efficient programme became unbounded over time. A uniform bound on the competitive prices, therefore, seems to be a promising candidate for the criterion we are trying to find. In fact, in a one-good model it is demonstrated by Peleg (1972) that an efficient programme is weakly value optimal if and only if it has bounded competitive prices. Thus, the second purpose of this paper is to obtain a generalized version of Peleg's result.

It is simple to show, in view of the related results in optimal growth theory (see McKenzie (1974)), that if an efficient programme has bounded competitive prices, then it is weakly value optimal. It turns out that the converse is also true if we restrict our attention to regular interior efficient programmes. So, Peleg's result generalizes to

Theorem 2. *Under* (A.1)–(A.4), *a regular interior efficient program* $(\bar{x}, \bar{y}, \bar{c})$ *is weakly value optimal if and only if it has a bounded discounted price sequence.*

(For a proof of Theorem 2, see Section 2).

2. PROOFS

Before coming to the proofs, we introduce some notation, and some facts from Mitra (1976) which will ease the writing. In the following facts (i) and (ii) can be found in Mitra ((1976), pp. 424–425), those in (iii) and (iv) in Mitra ((1976), pp. 426–427).

(i) Denote the sum vector $(1, \ldots, 1)$ in R^m by u. For an interior competitive programme $(\bar{x}, \bar{y}, \bar{c})$ [with $\bar{x}_t \geqq ku$ for $t \geqq 0$, $k > 0$], we denote $\bar{f}^t_{x^m}$ by r_t, $\bar{\pi}^m_t$ by R_t for $t \geqq 0$. We know that $\bar{\pi}^i_t = \bar{\pi}^m_t = R_t$ for $i = 1, \ldots, m-1$ and $t \geqq 0$, and so $\bar{p}_t = (\bar{q}_t/R_t)$ for $t \geqq 0$. For any feasible programme (x, y, c), $x_t \leqq Ku$, $y_{t+1} \leqq Ku$ for $t \geqq 0$, where K is given by (A.4). By (A.1)–(A.3), there are positive real numbers $a \leqq 1$, $\bar{a} < \infty$, such that for $\frac{1}{2}ku \leqq x \leqq Ku$, $\frac{1}{2}ku \leqq y \leqq Ku$, $a \leqq (-f_{y^i}) \leqq \bar{a}$ for $i = 1, \ldots, m-1$, and $a \leqq f_{x^i} \leqq \bar{a}$ for $i = 1, \ldots, m$. In particular, this means that $a \leqq r_t \leqq \bar{a}$, $au \leqq \bar{q}_t \leqq \bar{a}u$ for $t \geqq 0$. We denote $m\bar{a}K$ by A.

(ii) For (x, y) in the technology set, we denote $(\bar{p}_{t+1}\bar{y}_{t+1} - \bar{p}_t\bar{x}_t) - (\bar{p}_{t+1}y - \bar{p}_t x)$ by $w_t(x, y)$. By (1), $w_t(x, y) \geqq 0$ for $t \geqq 0$. If (x, y, c) is a feasible programme, we denote $\bar{q}_t(\bar{x}_t - x_t)$ by θ_t, and $\bar{p}_t(\bar{x}_t - x_t)$ by b_t. Then, for $t \geqq 1$, $\bar{p}_t(c_t - \bar{c}_t) = \bar{p}_t(y_t - \bar{y}_t) - \bar{p}_t(x_t - \bar{x}_t) = \bar{p}_t(\bar{x}_t - x_t) - \bar{p}_{t-1}(\bar{x}_{t-1} - x_{t-1}) - w_{t-1}(x_{t-1}, y_t)$. Thus,

$$\bar{p}_t(c_t - \bar{c}_t) = b_t - b_{t-1} - w_{t-1}(x_{t-1}, y_t). \tag{6}$$

Using (6) and $b_0 = 0$, we have for $T \geqq 1$,

$$\textstyle\sum_{t=1}^T p_t(c_t - \bar{c}_t) = b_T - \sum_{t=1}^T w_{t-1}(x_{t-1}, y_t). \tag{7}$$

(iii) There is a positive real number μ, such that if (x, y, c) is a feasible programme, with $x_t \geqq \frac{1}{2}ku$ for $t \geqq 0$, then $w_t(x_t, y_{t+1}) \geqq \mu\theta_t^2/R_{t+1}$ for $t \geqq 0$.

(iv) If (x', y', c') is a feasible programme, then by taking a $(\frac{1}{2}, \frac{1}{2})$ convex combination of the programmes (x', y', c') and $(\bar{x}, \bar{y}, \bar{c})$, we have a feasible programme (x, y, c) with $x_t \geqq \frac{1}{2}\bar{x}_t \geqq \frac{1}{2}ku$ for $t \geqq 0$, and

$$\textstyle\sum_{t=1}^T \bar{p}_t(c_t - \bar{c}_t) \geqq \frac{1}{2}\sum_{t=1}^T \bar{p}_t(c'_t - \bar{c}_t) \quad \text{for } T \geqq 1. \tag{8}$$

Proof of Theorem 1. (Sufficiency) Suppose an interior programme $(\bar{x}, \bar{y}, \bar{c})$ is weakly value maximal, but inefficient. Then there is a feasible program (x, y, c) such that $c_t \geqq \bar{c}_t$ for $t \geqq 1$, and $c_t > \bar{c}_t$ for some t. Since $\bar{p}_t \gg 0$ for $t \geqq 0$, (2) is violated. This contradiction proves sufficiency.

(Necessity) Suppose an interior programme $(\bar{x}, \bar{y}, \bar{c})$ is efficient, (hence competitive), but not weakly value maximal. By (2) and (iv), there is a feasible programme (x, y, c), a real number $z > 0$, and $T^* \geqq 1$, such that $x_t \geqq \frac{1}{2}ku$ for $t \geqq 0$, and

$$\textstyle\sum_{t=1}^T p_t(c_t - \bar{c}_t) \geqq z \quad \text{for } T \geqq T^*. \tag{9}$$

By (7) and (9), $b_T \geqq z$ for $T \geqq T^*$; so $\theta_T > 0$, and $\theta_T \leqq \bar{q}_T\bar{x}_T \leqq (\bar{a}u)(Ku) = m\bar{a}K = A$, for $T \geqq T^*$. By (7), (9) and (iii),

$$b_T \geqq z + \textstyle\sum_{t=1}^T w_{t-1}(x_{t-1}, y_t) \geqq z + \sum_{t=T^*}^{T-1} (\mu\theta_t^2/R_{t+1}) \tag{10}$$

for $T \geqq T^*+1$. Denoting the extreme right-hand expression in (10) by d_T, we note that for $T \geqq T^*+1$, $0 < d_T \leqq b_T$, and $d_{T+1} = d_T + (\mu\theta_T^2/R_{T+1}) = d_T + (\mu R_T^2 b_T^2/R_{T+1})$. So, for $T \geqq T^*+1$,

$$d_{T+1} \geqq d_T + (\mu R_T^2 d_T^2/R_{T+1}). \tag{11}$$

Denoting min $(1, \mu)$ by $\bar{\mu}$, and $R_{T+1}\, d_{T+1}\bar{\mu}$ by e_{T+1} for $T \geqq T^*$, and using (11), we have a sequence (e_t) satisfying (I) $0 < e_t \leqq A$ and (II) $e_{t+1} \geqq e_t(r_t + e_t)$ for $t \geqq T^*+1$. Then, following the method of Cass [1972, p. 219], it follows that the terms of trade of the m-th good must deteriorate too fast. Hence, by Mitra ((1976) Theorem 3.1), $(\bar{x}, \bar{y}, \bar{c})$ is inefficient, a contradiction. This establishes necessity. ||

Proof of Theorem 2. (Sufficiency) Suppose $(\bar{x}, \bar{y}, \bar{c})$ is an interior efficient (hence competitive) programme for which (5) holds, but which is not weakly value optimal. By (5), there is $0 < V < \infty$, such that $\bar{p}_t \leqq Vu$ for $t \geqq 0$. By (3) and (iv), there is a feasible programme (x, y, c), a real number $z > 0$, and a subsequence of periods, T_s, such that

$$\textstyle\sum_{t=1}^{T} p_t(c_t - \bar{c}_t) \geqq z \quad \text{for } T = T_s \tag{12}$$

and $x_t \geqq \frac{1}{2}ku$ for $t \geqq 0$. By (7) and (12), $b_T \geqq z$ for $T = T_s$. So, $\theta_T = R_T b_T \geqq (z/V)$, and $(\theta_T/R_{T+1}) = (R_T b_T/R_{T+1}) = (b_T/r_T) \geqq (z/\bar{a})$, for $T = T_s$. By (7), (8) and (iii), we obtain

$$b_T \geqq z + \textstyle\sum_{t=1}^{T} w_{t-1}(x_{t-1}, y_t) \geqq z + \sum_{t=1}^{T} (\mu\theta_t^2/R_{t+1}) \tag{13}$$

$\theta_t^2/R_{t+1} \geqq (z^2/V\bar{a})$ for $t = T_s$, and $\theta_t^2 \geqq 0$, for $t \neq T_s$, so

$$b_T \geqq z + (s-1)\mu(z^2/V\bar{a}) \quad \text{for } T = T_s. \tag{14}$$

Now, $b_T = (\theta_T/R_T) \leqq V\theta_T \leqq V\bar{q}_T\bar{x}_T \leqq VA$ for $T \geqq 0$. Using this in (14), we have a contradiction for large s, proving sufficiency.

(Necessity) Suppose a regular interior programme $(\bar{x}, \bar{y}, \bar{c})$ is efficient and weakly value optimal, but violates (5). Now, since $\bar{p}_t = \bar{q}_t/R_t$ for $t \geqq 0$, so $\bar{p}_t u \leqq [(\bar{a}u)u/R_t]$, using (i). So, there is a subsequence t_s, such that $R_{t_s} \to 0$ as $s \to \infty$.

Following the procedure in Mitra ((1976) p. 428) we can find a scalar $\bar{\theta} > 0$, and an output $\hat{y}_{t+1}^m \geqq \frac{1}{2}k$, such that $f({}^*\bar{y}_{t+1}; {}^*\bar{x}_t, \bar{x}_t^m - \bar{\theta}) = \hat{y}_{t+1}^m$. Let $d = \min_i [\inf_{t\geqq 1} \bar{c}_t^i]$ and $\theta = \min[(d/2\bar{a}), \bar{\theta}]$. Then, there is y_{t+1}^m, such that $y_{t+1}^m = f({}^*\bar{y}_{t+1}; {}^*\bar{x}_t, \bar{x}_t^m - \theta)$, and $y_{t+1}^m \geqq (\bar{y}_{t+1}^m - \frac{1}{2}d) \geqq (\bar{x}_{t+1}^m + \frac{1}{2}d)$.

Rename the sequence t_s as T. Then, find a subsequence T_n such that $(\theta/R_{T_{n+1}}) \geqq (\frac{1}{2}d/R_{T_n+1})$, and $T_{n+1} \geqq T_n + 2$. This is possible as $R_T \to 0$ as $T \to \infty$. Now construct a programme (x', y', c') in the following way: $x_t' = \bar{x}_t$ for $t \neq T_n$; $x_t' = \bar{x}_t - \theta$ for $t = T_n$; $y_{t+1}' = \bar{y}_{t+1}$ for $t \neq T_n$; $y_{t+1}' = ({}^*\bar{y}_{t+1}, y_{t+1}^m)$ for $t = T_n$; $c_t' = \bar{c}_t$ for $t \neq T_n$, $T_n + 1$; $c_t' = ({}^*\bar{c}_t, \bar{c}_t^m + \theta)$ for $t = T_n$; $c_t' = ({}^*\bar{c}_t, y_t^m - \bar{x}_t^m)$ for $t = T_n + 1$. It can be checked that, by construction, such a programme is feasible. By choice of the subsequence T_n, we also have

$$\theta\bar{p}_{T_1}^m \leqq \textstyle\sum_{t=1}^{T_n} \bar{p}_t(c_t' - \bar{c}_t). \tag{15}$$

It follows from (15) that $(\bar{x}, \bar{y}, c)$ is not weakly value optimal. This contradiction establishes necessity. ||

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